1 Grassmann manifold and Universal Bundle

1.1 Grassmann Manifold

1.1.1 Motivation

For each surface $M$ in $\mathbb{R}^3$, the Gauss map $\nu$ is a map from $M$ into $S^2$ by sending each point to its unit normal vector. If the surface is orientable, then the Gauss map induce a map from the surface into the complex projective space $\mathbb{P}^1$. More generally, suppose that $M$ is an $n$-dimensional smooth submanifold of $\mathbb{R}^{n+k}$. For each $x \in M$, the tangent space $T_xM$ can be identified with an $n$-dimensional subspace of $\mathbb{R}^{n+k}$. We may treat $T_xM$ as an element in so-called the Grassmannian manifold. The Grassmannian $G_n(\mathbb{R}^{n+k})$ is the set of all $n$-dimensional vector subspaces of $\mathbb{R}^{n+k}$. It is obvious that all of the tangent space of $M$ are points in the Grassmannian.

1.1.2 Stiefel Manifold and Grassmann Manifold and Their Topology

A $n$-frame in $\mathbb{R}^{n+k}$ is a point $v = (v_1, \cdots, v_n) \in (\mathbb{R}^{n+k})^n$ so that the set $\{v_1, \cdots, v_n\}$ is a set of linearly independent vectors in $\mathbb{R}^{n+k}$. The collection of all $n$-frames in $\mathbb{R}^{n+k}$, $V_n(\mathbb{R}^{n+k})$, forms an open subset of $(\mathbb{R}^{n+k})^n$ and is called the Stiefel manifold. There is a canonical map $q$ from the Stiefel manifold into the Grassmannian given by

$$q(v_1, \cdots, v_n) = \text{span}\{v_1, \cdots, v_n\}.$$ 

One can topologized $G_n(\mathbb{R}^{n+k})$ in the following way: we say that $O$ is open in $G_n(\mathbb{R}^{n+k})$ if and only if $q^{-1}(O)$ is open in $V_n(\mathbb{R}^{n+k})$. An orthonormal $k$-frame is a point $v = (v_1, \cdots, v_n)$ in $V_n(\mathbb{R}^{n+k})$ so that $v_i \cdot v_j = \delta_{ij}$ for any $i, j$. We shall denote the set of all orthonormal $k$-frame by $V_n^0(\mathbb{R}^{n+k})$ and the restriction of $q$ to $V_n^0(\mathbb{R}^{n+k})$ by $q_0$. It is clear that $V_n^0(\mathbb{R}^{n+k})$ is a compact subset of $(\mathbb{R}^{n+k})^n$ since it is closed and bounded. One can also topologize $G_n(\mathbb{R}^{n+k})$ in the following way: $O$ is open in $G_n(\mathbb{R}^{n+k})$ if and only if $q_0(O)$ is open in $V_n^0(\mathbb{R}^{n+k})$. We can show that these two topologies given by $q, q_0$ are the same. For each $v \in V_n(\mathbb{R}^{n+k})$, we can construct a point $G(v) \in V_n^0(\mathbb{R}^{n+k})$ via the Gram-Schmidt process. This construction $G: V_n(\mathbb{R}^{n+k}) \to V_n^0(\mathbb{R}^{n+k})$ is obviously continuous. Suppose $T$ and $T_0$ are the two topologies on $G_n(\mathbb{R}^{n+k})$ given by $q, q_0$ respectively. Suppose $O$ is $T_0$-open, then $q_0^{-1}(O)$ is open in $V_n^0(\mathbb{R}^{n+k})$ and thus $q^{-1}(O) = G^{-1}(q_0^{-1}(O))$ is open in $V_n(\mathbb{R}^{n+k})$. Therefore $O$ is $T$ open. Now let us consider the inclusion map $i: V_n^0(\mathbb{R}^{n+k}) \to V_n(\mathbb{R}^{n+k})$. We know due to the relative topology of $V_n^0(\mathbb{R}^{n+k})$, the inclusion map is continuous. Hence for each $T$-open set $O$ in $G_n(\mathbb{R}^{n+k})$, $q_0^{-1}(O) = i^{-1}(q^{-1}(O))$ is open in $V_n^0(\mathbb{R}^{n+k})$. As a consequence, $O$ is $T_0$-open. From these observation, we find that $G_n(\mathbb{R}^{n+k})$ is a compact space since it is the image of a compact space under the continuous map $q_0$.

For each $w \in \mathbb{R}^{n+k}$ and $X \in G_n(\mathbb{R}^{n+k})$, define

$$\rho_w(X) = d(w, X)^2.$$ 

Consider the map $V_n^0(\mathbb{R}^{n+k})$ given by

$$f(v_1, \cdots, v_n) = w \cdot w - \sum_{k=1}^{n}(w \cdot v_k)^2.$$ 

Then it is obvious that $f$ is continuous and $f = \rho_w \circ q_0$. We find that for any $O$ open in $\mathbb{R}$, $q_0^{-1}(\rho_w^{-1}(O)) = f^{-1}(O)$ is open, i.e. $\rho_w^{-1}(O)$ is open in $G_n(\mathbb{R}^{n+k})$. Hence the map $\rho_w: G_n(\mathbb{R}^{n+k}) \to \mathbb{R}$ is continuous. Suppose $X \neq Y$ are two points in $G_n(\mathbb{R}^{n+k})$. Choose $w \in X$ but $w \notin Y$. Then $\rho_w(X) = 0 \neq \rho_w(Y)$. Hence $\rho_w$ is a continuous function separating $X$ and $Y$. This tells us that $G_n(\mathbb{R}^{n+k})$ is a Hausdorff space.

1.1.3 Grassmann Manifold as a Topological Manifold

Let $X_0$ be any point in $G_n(\mathbb{R}^{n+k})$. We know $\mathbb{R}^{n+k} = X_0 \oplus X_0^\perp$. Let $P_0: \mathbb{R}^{n+k} \to X_0$ be the orthogonal projection onto $X_0$. For each $X \in G_n(\mathbb{R}^{n+k})$, we might consider $P_0^X: X \to X_0$ the restriction of $P_0$ to $X$. Let $U$ be the set of all points $Y$ in $G_n(\mathbb{R}^{n+k})$ so that $Y \cap X_0^\perp = \{0\}$. Then one can show that $U$ is an open set in $G_n(\mathbb{R}^{n+k})$ and for each $X \in U$, $P_0^X: X \to X_0$ is an isomorphism. Hence for each $x \in X_0$, $(P_0^X)^{-1}x \in X$. Define a map $K^X: X_0 \to \mathbb{R}^{n+k}$ by $K^X x = (P_0^X)^{-1}x - x$. If we denote $x' = K^X x$, then $P_0^X(x + x') = x$ would imply that $x' \in X_0^\perp$. Hence $K^X: X_0 \to X_0^\perp$. Conversely for each $K \in L(X_0, X_0^\perp)$,
define \( X = \{ x + Kx : x \in X_0 \} \). Then we find that \( P^X : X \to X_0 \) by \( P^X(x + Kx) = x \) defines an isomorphism and therefore \( X \cap X_0^\perp = \{ 0 \} \), i.e. \( X \subseteq U \). Moreover, we also have \( K = K^X \). This establish an one-to-one correspondence between \( U \) and \( L(X_0, X_0^\perp) \) defined by \( X \to K^X \). Since we can identify the space \( L(X_0, X_0^\perp) \) with \( \mathbb{R}^{nk} \) by choosing a suitable basis for \( X_0 \) and \( X_0^\perp \). Let us fix an orthonormal basis \( \{ x_1, \ldots, x_n \} \) for \( X_0 \).

Then for any \( Y \) in \( U \), there exists a unique basis \( \{ x_1^Y, \ldots, x_n^Y \} \) so that

\[
p^Y(x_i^Y) = x_i, \quad 1 \leq i \leq n.
\]

From here, we also know that for any \( 1 \leq i \leq n \),

\[
x_i^Y = x_i + K^Y x_i.
\]

Define a map \( F : U \to V_n(\mathbb{R}^{n+k}) \) by \( Y \mapsto (x_1^Y, \ldots, x_n^Y) \). Then this map is continuous. Hence \( \{ x_i^Y \} \) depends continuously on \( Y \) and so is \( K^Y \) or one can This implies that the one-to-one correspondence between \( U \) and \( L(X_0, X_0^\perp) \) is in fact a homeomorphism. This shows that \( G_n(\mathbb{R}^{n+k}) \) is a compact Hausdorff topological manifold of dimension \( nk \). Next, we want to show that the map \( G_n(\mathbb{R}^{n+k}) \to G_k(\mathbb{R}^{n+k}) \) given by \( X \mapsto X^\perp \) is homeomorphism. Since we have already showen that the Grassmannian is a topological manifold, to show \( \downarrow : X \to X^\perp \) is continuous, we only need to show that in each neighborhood \( U \) defined above, \( \downarrow \) is a continuous function.

Let us fixed a basis \( \{ \pi_1, \ldots, \pi_k \} \) in \( X_0^\perp \). Define a function \( h : q^{-1}(U) \to V_n(\mathbb{R}^{n+k}) \) as follows. For each \( (y_1, \ldots, y_n) \in q^{-1}(U) \), define \( h(y_1, \ldots, y_n) = G(y_1, \ldots, y_n, \pi_1, \ldots, \pi_n) \). Here \( G \) is the Gram-Schmidt operator on \( V_n(\mathbb{R}^{n+k}) \). It is obvious that \( h \) is a continuous function. If we denote \( h(y_1, \ldots, y_n) = (y_1', \ldots, y_n', y_{n+1}', \ldots, y_{n+k}') \), then \( \{ y_i' : 1 \leq i \leq n \} \) and \( \{ y_{n+i}' : 1 \leq i \leq k \} \) form basis for \( Y \) and \( Y^\perp \) respectively. Define \( h' : q^{-1}(U) \to V_k(\mathbb{R}^{n+k}) \) by \( h'(y_1', \ldots, y_n') = (y_{n+1}', \ldots, y_{n+k}') \). Then \( h' \) is a continuous map (it is the composition of \( h \) with a projection operator). Then we obtain the following commutative diagram:

\[
\begin{array}{ccc}
q^{-1}(U) \subset V_n(\mathbb{R}^{n+k}) & \xrightarrow{h'} & V_k(\mathbb{R}^{n+k}) \\
\downarrow q \downarrow & & \downarrow q \\
U \subset G_n(\mathbb{R}^{n+k}) & \xrightarrow{\downarrow} & G_k(\mathbb{R}^{n+k}).
\end{array}
\]

Hence \( \downarrow : X \to X^\perp \) is a continuous function. Now we conclude that:

**Lemma 1.1.** The Grassmann Manifold \( G_n(\mathbb{R}^{n+k}) \) is a compact topological manifold of dimension \( nk \). The correspondence \( X \mapsto X^\perp \) which assigns to each \( n \)-vector subspace of \( \mathbb{R}^{n+k} \) its orthogonal complement defines a homeomorphism between \( G_n(\mathbb{R}^{n+k}) \) and \( G_k(\mathbb{R}^{n+k}) \).

### 1.1.4 Exercise

1. Let \( M \) be any vector subspace of \( \mathbb{R}^n \) and \( w \in \mathbb{R}^n \) be any point. Define

\[
d(w, M) = \inf_{m \in M} d(w, m).
\]

(a) Show that \( M \) is a closed subset of \( \mathbb{R}^n \).

(b) Show that there exists \( m_0 \in M \) so that \( d(w, M) = d(w, m_0) \).

(c) Show that \( w - m_0 \perp M \).

(d) Define \( f : \mathbb{R}^n \to \mathbb{R} \) by \( f(w) = d(w, M) \). Show that \( f \) is a continuous map.

2. Show that \( V_n(\mathbb{R}^{n+k}) \) is an open subset in \( \mathbb{R}^{n+k} \).

3. Suppose \( X_0 \) is a given point in \( G_n(\mathbb{R}^{n+k}) \) and \( U \) be the set of all points \( X \) of \( G_n(\mathbb{R}^{n+k}) \) so that \( X \cap X_0^\perp = \{ 0 \} \).

(a) Show that \( U \) is an open subset of \( G_n(\mathbb{R}^{n+k}) \).

(b) For each \( X \in U \), define \( P^X \) to be the restriction of the projection \( P : \mathbb{R}^{n+k} \to X_0 \) to \( X \). Given a basis \( \{ x_i : 1 \leq i \leq n \} \), there exists a basis \( \{ x_i^X : 1 \leq i \leq n \} \) of \( X \) so that \( p^X(x_i^X) = x_i \). Define a map \( F : U \to V_n(\mathbb{R}^{n+k}) \) by \( F(X) = (x_1^X, \ldots, x_n^X) \). Show that \( F \) is continuous.
4. For each ordered linearly independent set \( \{v_1, \cdots, v_n\} \) of \( \mathbb{R}^{n+k} \), we can obtained an ordered orthonormal set \( \{y_1, \cdots, y_n\} \) by using the Gram-Schmidt Process to \( \{v_1, \cdots, v_n\} \). Define \( G(v_1, \cdots, v_n) = (y_1, \cdots, y_n) \). Show that \( G : V_n(\mathbb{R}^{n+k}) \to V^0_n(\mathbb{R}^{n+k}) \) is a continuous map.

5. Suppose \( X \) is a topological space. Show that if for any \( x \neq y \) in \( X \), there exists a continuous function \( f : X \to \mathbb{R} \) so that \( f(x) \neq f(y) \), then \( X \) is a Hausdorff space.

6. Let \( X \) be a topological space and \( R \) is an equivalence relation on it. Denote \( X/R \) the set of all equivalent classes of \( X \) with respect to \( R \). Define \( \pi : X \to X/R \) by sending any point of \( x \) into its equivalent class.

   (a) On \( X/R \), we define \( O \in T \) if and only if \( \pi^{-1}(O) \) is open. Show that \( T \) is a topology on \( X/R \). Note that this topology is called the quotient topology.

   (b) Suppose \( f : X/R \to Y \) is a map. Show that \( f \) is continuous if and only if for any \( F : X \to Y \) so that \( F = \pi \circ f \), \( F \) is a continuous function.